

# HJB equations in infinite dimensions under weak regularizing properties

Federica Masiero

Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca  
via Cozzi 55, 20125 Milano, Italy  
e-mail: federica.masiero@unimib.it

## Abstract

We solve in mild sense Hamilton Jacobi Bellman equations, both in an infinite dimensional Hilbert space and in a Banach space, with lipschitz Hamiltonian and lipschitz continuous final condition, and asking only a weak regularizing property on the transition semigroup of the corresponding state equation. The results are applied to solve stochastic optimal control problems; the models we can treat include a controlled stochastic heat equation in space dimension one and with control and noise on a subdomain.

## 1 Introduction

In this paper we study semilinear Kolmogorov equations in an infinite dimensional Hilbert space  $H$ , as well as in a Banach space  $E$ , in particular Hamilton Jacobi Bellman equations. More precisely, let us consider the following equation

$$\begin{cases} -\frac{\partial v}{\partial t}(t, x) = \mathcal{L}_t v(t, x) + \psi(\nabla v(t, x) B) + l(t, x), & t \in [0, T], x \in H \text{ or } x \in E \\ v(T, x) = \phi(x). \end{cases} \quad (1.1)$$

The second order differential operator  $\mathcal{L}_t$  is the generator of the transition semigroup  $P_t$  related to the following perturbed Ornstein-Uhlenbeck process

$$\begin{cases} dX_t = AX_t dt + BF(t, X_t) + BdW_t, & t \in [0, T] \\ X_0 = x, \end{cases} \quad (1.2)$$

that is, at least formally,

$$(\mathcal{L}_t f)(x) = \frac{1}{2}(\text{Tr} Q \nabla^2 f)(x) + \langle Ax + F(t, x), \nabla f(x) \rangle.$$

For the sake of simplicity the paper focuses on the case of an Ornstein-Uhlenbeck process, that is  $F = 0$  in equation (1.2), and hints on how to handle the case of a perturbed Ornstein-Uhlenbeck process are given throughout the paper when necessary. From now on also in the introduction we assume  $F = 0$ . We also stress the fact that in this paper we aim to reduce the technical difficulties: with some efforts the case of Kolmogorov equations with non linear term given by  $\hat{\psi}(t, x, \nabla v(t, x) B)$  instead of  $\psi(\nabla v(t, x) B) + l(t, x)$  can be treated, but in the present paper we study only the special case of semilinear Kolmogorov equations like (1.1).

Second order differential equations on Hilbert spaces have been extensively studied: see e.g. the monograph [5]. One of the main motivations for this study in the non linear case is the

connection with control theory, namely the fact that in many cases the value function of a stochastic optimal control problem is a solution to such an equation.

Solutions of semilinear Kolmogorov equations (1.1) are studied in the literature both by an analytic approach and by a purely probabilistic approach. In the first direction we mention the paper [10], where the main assumption is a regularizing property for the transition semigroup  $P_t$ , namely the strong Feller property. In the same direction we cite also the papers [13] and [14] where equation with the special structure of (1.1) is considered in an Hilbert space and in a Banach space respectively, by requiring on the transition semigroup a regularizing property strictly weaker than the Strong Feller one.

For what concerns the purely probabilistic approach, semilinear Kolmogorov equations in  $H$  with the special structure of (1.1) and with non constant  $B$ , are treated in the paper [8] by means of backward stochastic differential equations (BSDEs in the following). No regularizing assumption on the transition semigroup is imposed, on the contrary  $\psi$ ,  $l$  and  $\phi$  are assumed differentiable. The papers [15] and [16] are the extension of [8] to the Banach space framework, with some restrictions on  $B$ , which is asked to be constant. We notice that [8] is the infinite dimensional extension of results in [19].

In the present paper we use both the probabilistic and the analytic approach, since we want to treat the case of  $\psi$ ,  $l(t, \cdot)$  and  $\phi$  only Lipschitz continuous by requiring a directional regularizing property on the transition semigroup  $P_t$ , similar but weaker than the one in [13] and [14]. The BSDE approach in the case of Lipschitz continuous data has been used in [9], in that paper the solution of the HJB equation is given in a sense weaker than the mild sense, since the directional derivative is addressed as a generalized gradient. So the present paper improves the results in [9] in the situations when the following assumptions in (1.3) and (1.4) are satisfied.

Coming into more details, we assume that it holds true

$$\left\| Q_t^{-1/2} e^{tA} B \right\| \leq c(t), \text{ for } 0 < t \leq T \quad (1.3)$$

where

$$Q_t = \int_0^t e^{sA} B B^* e^{sA^*} ds$$

As a consequence we have a regularizing property for the transition semigroup  $P_t$ : it maps bounded and continuous functions into  $B$ -Gâteaux differentiable functions and for every bounded and continuous function  $\phi$

$$|\nabla^B P_\tau [\phi](x) \xi| \leq c(\tau) \|\phi\|_\infty |\xi|. \quad (1.4)$$

The novelty of this paper towards [13] and [14] is the fact that  $c$ , that it is expected to blow up as  $\tau$  goes to 0, may be not locally integrable at 0.

In order to prove existence and uniqueness of a mild solution  $v$  of equation (1.1), we use the fact that if  $l$ ,  $\psi$  and  $\phi$  are Gâteaux differentiable,  $v$  can be represented in terms of the solution of a suitable forward-backward system (FBSDE in the following):

$$\begin{cases} dX_\tau = AX_\tau d\tau + BdW_\tau, & \tau \in [t, T] \subset [0, T], \\ X_t = x, \\ dY_\tau = -\psi(Z_\tau) d\tau - l(\tau, X_\tau) d\tau + Z_\tau dW_\tau, \\ Y_T = \phi(X_T). \end{cases} \quad (1.5)$$

It is well known, see e.g. [19] for the finite dimensional case and [8] and [15] for the generalization to the infinite dimensional case, respectively in the Hilbert space framework and in the Banach space framework, that, letting  $v$  be solution of the Kolmogorov equation (1.1) when all the data  $l$ ,  $\psi$  and  $\phi$  are differentiable,  $v(t, x) = Y_t^{t,x}$  and  $\nabla v(t, x)B = Z_t^{t,x}$ . This identification has been

extended in [13] to the case of data continuous and with the transition semigroup satisfying the regularizing property stated in (1.4), with  $c(\cdot)$  locally integrable. These identifications are here extended, both in Hilbert and in the Banach space case, to the case of Lipschitz continuous coefficients with the transition semigroup satisfying the regularizing property in (1.4), with  $c(\cdot)$  not locally integrable.

The model we have in mind is a stochastic heat equation on the space interval  $[0, 1]$ , and with noise on a subdomain, whose closure is strictly contained in  $[0, 1]$ , see Section 2 for more details. This equation can be reformulated as an evolution equation both in the Hilbert space  $H$  of square integrable functions on  $[0, 1]$  and in the Banach space of continuous functions on  $[0, 1]$ . We notice that for such a stochastic partial differential equations, suitably reformulated in an infinite dimensional space of functions, conditions (1.3) and (1.4) hold true, with  $c(t) \sim e^{\frac{1}{t}}$ , as  $t \rightarrow 0$ , see the papers [7], [22] and [23].

The paper is organized as follows: in Section 2 some results on the Ornstein-Uhlenbeck process in  $H$  are collected and the stochastic heat equation we can treat is presented, in Section 3 we present the optimal control problem we can treat in the Hilbert space framework, in Section 4 the Kolmogorov equation (1.1) is solved in the Hilbert space  $H$ . Then we turn to the Banach space case: the choice of presenting first the Hilbert space case and then turn to the Banach space case is due to our willing of presenting the results in the simplest context, and then to pass to more complicated situations. In Section 5 we present the Ornstein-Uhlenbeck process in the Banach space  $E$  and we solve in  $E$  the Kolmogorov equation (1.1), finally in Section 6 we solve the optimal control problem in the Banach space  $E$ : we notice that we perform the fundamental relation in a Banach space without differentiability assumptions on the costs and on the Hamiltonian, and this is an improvement towards [15]. Finally we show how our results apply to a controlled stochastic heat equation in  $E$  with control and noise on a subdomain.

## 2 Preliminary results on the forward equation and its semigroup

We consider an Ornstein-Uhlenbeck process in a real and separable Hilbert space  $H$ , that is a Markov process  $X$  (also denoted  $X^{t,x}$  to stress the dependence on the initial conditions) solution to equation

$$\begin{cases} dX_\tau = AX_\tau d\tau + BdW_\tau, & \tau \in [t, T] \\ X_t = x, \end{cases} \quad (2.1)$$

where  $A$  is the generator of a strongly continuous semigroup in  $H$  and  $B$  is a linear bounded operator from  $\Xi$  to  $H$ . We define a positive and symmetric operator

$$Q_\sigma = \int_0^\sigma e^{sA} B B^* e^{sA^*} ds.$$

Throughout the paper we assume the following.

**Hypothesis 2.1** 1. *The linear operator  $A$  is the generator of a strongly continuous semigroup  $(e^{tA}, t \geq 0)$  in the Hilbert space  $H$ . It is well known that there exist  $M > 0$  and  $\omega \in \mathbb{R}$  such that  $\|e^{tA}\|_{L(H,H)} \leq M e^{\omega t}$ , for all  $t \geq 0$ . In the following, we always consider  $M \geq 1$  and  $\omega \geq 0$ .*

2.  *$B$  is a bounded linear operator from  $\Xi$  to  $H$  and  $Q_\sigma$  is of trace class for every  $\sigma \geq 0$ .*

The process  $X^{t,x}$  is clearly time-homogeneous, and for  $0 \leq t \leq \tau \leq T$  we denote by  $P_{\tau-t} = P_{t,\tau}$  its transition semigroup, where for every bounded and continuous function  $\phi : H \rightarrow \mathbb{R}$

$$P_{t,\tau}[\phi](x) = \mathbb{E}\phi(X_\tau^{t,x}).$$

It is well known that the Ornstein-Uhlenbeck semigroup can be represented as

$$P_\tau[\phi](x) := \int_H \phi(y) \mathcal{N}(e^{\tau A}x, Q_\tau)(dy), \quad \tau > 0,$$

and  $\mathcal{N}(e^{\tau A}x, Q_\tau)(dy)$  denotes a Gaussian measure with mean  $e^{\tau A}x$ , and covariance operator  $Q_\tau$ .

We briefly introduce the notion of  $B$ -differentiability, see e.g. [13]. We recall that for a continuous function  $f : H \rightarrow \mathbb{R}$  the  $B$ -directional derivative  $\nabla^B f$  at a point  $x \in H$  in direction  $\xi \in H$  is defined as follows:

$$\nabla^B f(x; \xi) = \lim_{s \rightarrow 0} \frac{f(x + sB\xi) - f(x)}{s}, \quad s \in \mathbb{R}.$$

A continuous function  $f$  is  $B$ -Gâteaux differentiable at a point  $x \in H$  if  $f$  admits the  $B$ -directional derivative  $\nabla^B f(x; \xi)$  in every directions  $\xi \in \Xi$  and there exists a functional, the  $B$ -gradient  $\nabla^B f(x) \in \Xi^*$  such that  $\nabla^B f(x; \xi) = \nabla^B f(x) \xi$ .

Throughout the paper we assume the following:

**Hypothesis 2.2** *The operators  $A$  and  $B$  are such that*

$$\text{Im } e^{tA}B \subset \text{Im } Q_t^{1/2}. \quad (2.2)$$

*As an immediate consequence it turns out the operator  $Q_t^{-1/2}e^{tA}B$  is well defined. Assume that there exists  $c : (0, T] \rightarrow \mathbb{R}$ , such that  $c$  is not integrable in 0 and  $c$  is bounded on any interval  $I \in [0, T]$ , such that  $0 \notin \bar{I}$ , where  $\bar{I}$  is the closure of  $I$ , moreover we ask that  $c(\cdot)$  is monotone not increasing in the interval  $(0, T]$ . Assume that it holds true*

$$\|Q_t^{-1/2}e^{tA}B\| \leq c(t), \text{ for } 0 < t \leq T. \quad (2.3)$$

As a consequence we have a regularizing property for the transition semigroup  $P_t$ : it maps bounded and continuous functions to  $B$ -Gâteaux differentiable functions. In the following we will refer to this regularizing property as  $B$ -regularizing property.

**Lemma 2.3** *Let  $A$  and  $B$  satisfy hypothesis 2.1. For some  $c : (0, T] \rightarrow \mathbb{R}$  measurable and for every  $\phi \in C_b(H)$ , the function  $P_\tau[\phi](x)$  is  $B$ -differentiable with respect to  $x$ , for every  $0 \leq t < \tau < T$  and for every  $\xi \in \Xi$ , and for  $0 \leq t < \tau \leq T$ ,*

$$|\nabla^B P_\tau[\phi](x) \xi| \leq c(\tau) \|\phi\|_\infty |\xi|. \quad (2.4)$$

*then hypothesis 2.3 is satisfied.*

**Proof.** The proof goes on like the proof of lemma 3.4 in [13], where  $c(t) = t^{-\alpha}$ ,  $0 < \alpha < 1$ . Note that in the present paper we are mainly concerned with the case  $c(\cdot) \notin L^1([0, T])$ , but this aspect does not enter the proof of the present result. The case  $c(\cdot) \in L^1([0, T])$  is treatable as in [13]. Note also that the  $B$ -regularizing property for the transition semigroup  $P_t$  stated in (2.4) is equivalent to the inclusion in (2.2).  $\square$

**Remark 2.4** *In the case of  $A$  and  $B$  satisfying hypothesis 2.2 the transition semigroup of the perturbed Ornstein Uhlenbeck process*

$$\begin{cases} dX_\tau = AX_\tau d\tau + BF(\tau, X_\tau) + BdW_\tau, & \tau \in [t, T] \\ X_t = x, \end{cases} \quad (2.5)$$

*satisfies the  $B$ -regularizing property proved in lemma 2.3, and stated in (2.4), by assuming that  $F$  is jointly continuous in  $t$  and  $x$  and lipschitz continuous in  $x$  uniformly with respect to  $t$  and Gâteaux differentiable with respect to  $x$ . For the proof see [14], Section 4, Theorem 4.3.*

The model we have in mind is a semilinear heat equation. Namely let  $\mathcal{O}$  be a subinterval of the interval  $[0, 1]$  such that  $\bar{\mathcal{O}}_0 \subsetneq [0, 1]$ . In the following  $\mathcal{O} = [a, b]$ ,  $0 < a < b < 1$ . We denote by  $H$  the Hilbert space  $L^2([0, 1])$  and the equation

$$\begin{cases} \frac{\partial y}{\partial s}(s, \xi) = \Delta y(s, \xi) + 1_{\mathcal{O}}(\xi) f(s, y(s, \xi)) + 1_{\mathcal{O}}(\xi) \frac{\partial W}{\partial s}(s, \xi), & s \in [t, T], \xi \in [0, 1], \\ y(t, \xi) = x(\xi), \\ \frac{\partial}{\partial \xi} y(s, \xi) = 0, & \xi = 0, \xi = 1. \end{cases} \quad (2.6)$$

Here  $\frac{\partial W}{\partial s}(s, \xi)$  is a space time white noise and  $f : [0, T] \times [0, 1] \rightarrow \mathbb{R}$  is a continuous function such that  $f(s, \cdot) : [0, 1] \rightarrow \mathbb{R}$  is differentiable with derivative uniformly bounded with respect to  $s \in [0, T]$ . Let  $F$  be the evaluation operator associated to  $f$  and  $B$  the multiplication operator associated to  $1_{\mathcal{O}}$ : for every  $h \in H$ ,  $Bh(\xi) = 1_{\mathcal{O}}(\xi)h(\xi)$ . With this definition of  $F$  and  $B$ , equation (2.6) can be written in an abstract way in  $H$  as a perturbed Ornstein Uhlenbeck process, see equation (2.5), where  $A$  is the Laplace operator with Neumann boundary conditions and  $W$  is a cylindrical Wiener process in  $H$ .

### 3 The optimal control problem

We consider the following controlled state equation

$$\begin{cases} dX_{\tau}^u = [AX_{\tau}^u + Bu_{\tau}] d\tau + BdW_{\tau}, & \tau \in [t, T] \\ X_t^u = x. \end{cases} \quad (3.1)$$

The solution of this equation will be denoted by  $X_{\tau}^{u,t,x}$  or simply by  $X_{\tau}^u$ .  $X$  is also called the state,  $T > 0$ ,  $t \in [0, T]$  are fixed. The process  $u$  represents the control and it is an  $(\mathcal{F}_{\tau})_{\tau}$ -predictable process with values in a closed and bounded set  $U$  of the Hilbert space  $\Xi$ , such that  $|u| \leq R$ . The occurrence of the operator  $B$  in the control term is imposed by our techniques and, among other facts, it allows to study the optimal control problem related by means of BSDEs.

Beside equation (3.1), define the cost

$$J(t, x, u) = \mathbb{E} \int_t^T [l(s, X_s^u) + g(u_s)] ds + \mathbb{E} \phi(X_T^u). \quad (3.2)$$

for real functions  $l$  on  $[0, T] \times H$ ,  $\phi$  on  $H$  and  $g$  on  $U$ . The control problem in strong formulation is to minimize this functional  $J$  over all admissible controls  $u$ . We make the following assumptions on the cost  $J$ .

**Hypothesis 3.1** 1. The function  $\phi : H \rightarrow \mathbb{R}$  is bounded and lipschitz continuous;

2.  $l : [0, T] \times H \rightarrow \mathbb{R}$  is bounded, lipschitz continuous with respect to  $x \in H$  uniformly with respect to  $t \in [0, T]$ ;

3.  $g : U \rightarrow \mathbb{R}$  is bounded and continuous.

We deduce immediately that equation (3.1) admits a unique mild solution, for every admissible control  $u$ .

We denote by  $J^*(t, x) = \inf_{u \in \mathcal{A}_d} J(t, x, u)$  the value function of the problem and, if it exists, by  $u^*$  the control realizing the infimum, which is called optimal control.

We define in a classical way the Hamiltonian function relative to the above problem:

$$\psi(z) = \inf_{u \in U} \{g(u) + zu\} \quad \forall z \in \Xi. \quad (3.3)$$

By our assumptions the Hamiltonian function is lipschitz continuous.

We define

$$\Gamma(z) = \{u \in U : zu + g(u) = \psi(z)\}; \quad (3.4)$$

if  $\Gamma(z) \neq \emptyset$  for every  $z \in \Xi$ , by [1], see Theorems 8.2.10 and 8.2.11,  $\Gamma$  admits a measurable selection, i.e. there exists a measurable function  $\gamma : H \rightarrow U$  with  $\gamma(z) \in \Gamma(z)$  for every  $z \in \Xi$ .

## 4 The semilinear Kolmogorov equation

The aim of this section is to present existence and uniqueness results for the solution of the Hamilton Jacobi Bellman equation ( HJB in the following ) related to the optimal control problem presented in section 3.

More precisely, let  $\mathcal{L}$  be the generator of the transition semigroup  $P_t$ , that is, at least formally,

$$(\mathcal{L}f)(x) = \frac{1}{2}(Tr BB^* \nabla^2 f)(x) + \langle Ax, \nabla f(x) \rangle.$$

Let us consider the following equation

$$\begin{cases} -\frac{\partial v}{\partial t}(t, x) = \mathcal{L}v(t, x) + \psi(\nabla^B v(t, x)) + l(t, x), & t \in [0, T], x \in H \\ v(T, x) = \phi(x), \end{cases} \quad (4.1)$$

We introduce the notion of mild solution of the non linear Kolmogorov equation (4.1), see e.g. [8] and also [13] for the definition of mild solution when  $\psi$  depends only on  $\nabla^B v$  and not on  $\nabla v$ . Since  $\mathcal{L}$  is (formally) the generator of  $P_t$ , the variation of constants formula for (4.1) is:

$$v(t, x) = P_{t,T}[\phi](x) + \int_t^T P_{t,s}[\psi(\nabla^B v(s, \cdot))](x) ds + \int_t^T P_{t,s}[l(s, \cdot)](x) ds, \quad t \in [0, T], x \in H. \quad (4.2)$$

We use this formula to give the notion of mild solution for the non linear Kolmogorov equation (4.1); we have also to introduce some spaces of continuous functions, where we seek the solution of (4.1).

As stated before we focus on the case  $c(\cdot) \notin L^1([0, T])$ : let  $C_{c(\cdot)}([0, T] \times H)$  be the linear space of continuous functions  $f : [0, T] \times H \rightarrow \mathbb{R}$  such that

$$\sup_{t \in [0, T]} \sup_{x \in H} (c(T-t))^{-1} |f(t, x)| < +\infty.$$

$C_{c(\cdot)}([0, T] \times H)$  endowed with the norm

$$\|f\|_{C_{c(\cdot)}} = \sup_{t \in [0, T]} \sup_{x \in H} (c(T-t))^{-1} |f(t, x)|,$$

is a Banach space.

We consider also the linear space  $C_{c(\cdot)}^s([0, T] \times H, \Xi^*)$  of the mappings  $L : [0, T] \times H \rightarrow \Xi^*$  such that for every  $\xi \in \Xi$ ,  $L(\cdot, \cdot)\xi \in C_{c(\cdot)}([0, T] \times H)$ . The space  $C_{c(\cdot)}^s([0, T] \times H, \Xi^*)$  turns out to be a Banach space if it is endowed with the norm

$$\|L\|_{C_{c(\cdot)}(\Xi^*)} = \sup_{t \in [0, T]} \sup_{x \in H} (c(T-t))^{-1} \|L(t, x)\|_{\Xi^*}.$$

In other words,  $C_{c(\cdot)}^s([0, T] \times H, \Xi^*)$  can be identified with the space of the operators  $L(H, C_{c(\cdot)}([0, T] \times H))$ .

**Definition 4.1** We say that a function  $v : [0, T] \times H \rightarrow \mathbb{R}$  is a mild solution of the non linear Kolmogorov equation (4.1) if the following are satisfied:

1.  $v \in C_b([0, T] \times H)$ ;
2.  $\nabla^B v \in C_{c(\cdot)}^s([0, T] \times H, \Xi^*)$ : in particular this means that for every  $t \in [0, T)$ ,  $v(t, \cdot)$  is  $B$ -differentiable;
3. equality (4.2) holds.

Existence and uniqueness of a mild solution of equation (4.1) are related to the study of the following forward-backward system: for given  $t \in [0, T]$  and  $x \in H$ ,

$$\begin{cases} dX_\tau = AX_\tau d\tau + BdW_\tau, & \tau \in [t, T] \subset [0, T], \\ X_t = x, \\ dY_\tau = -\psi(Z_\tau) d\tau - l(\tau, X_\tau) d\tau + Z_\tau dW_\tau, \\ Y_T = \phi(X_T), \end{cases} \quad (4.3)$$

and to the identification of  $Z_t^{t,x} = \nabla_x Y_t^{t,x} B$ . We extend the definition of  $X$  setting  $X_s = x$  for  $0 \leq s \leq t$ . The second equation in (4.3), namely

$$\begin{cases} dY_\tau = -\psi(Z_\tau) d\tau - l(\tau, X_\tau) d\tau + Z_\tau dW_\tau, & \tau \in [0, T], \\ Y_T = \phi(X_T), \end{cases} \quad (4.4)$$

is of backward type. Under suitable assumptions on the coefficients  $\psi : \Xi \rightarrow \mathbb{R}$ ,  $l : [0, T] \times H \rightarrow \mathbb{R}$  and  $\phi : H \rightarrow \mathbb{R}$  we will look for a solution consisting of a pair of predictable processes, taking values in  $\mathbb{R} \times H$ , such that  $Y$  has continuous paths and

$$\|(Y, Z)\|_{\mathbb{K}_{cont}}^2 := \mathbb{E} \sup_{\tau \in [0, T]} |Y_\tau|^2 + \mathbb{E} \int_0^T |Z_\tau|^2 d\tau < \infty,$$

see e.g. [18]. In the following we denote by  $\mathbb{K}_{cont}([0, T])$  the space of such processes.

The solution of (4.3) will be denoted by  $(X_\tau, Y_\tau, Z_\tau)_{\tau \in [0, T]}$ , or, to stress the dependence on the initial time  $t$  and on the initial datum  $x$ , by  $(X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x})_{\tau \in [0, T]}$ . In the following we refer to [8] for the definition of the class  $\mathcal{G}(H)$  of Gâteaux differentiable functions  $f : H \rightarrow \mathbb{R}$  with strongly continuous derivative. We make differentiability assumptions on the coefficients that we are going to remove in the sequel.

**Hypothesis 4.1** The map  $\psi : \Xi \rightarrow \mathbb{R}$  is in  $\mathcal{G}(\Xi)$ , and the maps  $l(t, \cdot) : H \rightarrow \mathbb{R}$  and  $\phi : H \rightarrow \mathbb{R}$  belong to  $\mathcal{G}(H)$ .

Note that since the Hamiltonian function is lipschitz continuous and by lipschitz assumptions on  $l$  and  $\phi$ , see hypothesis 3.1, it turns out that  $\nabla \psi(z)$ ,  $\nabla_x l(t, x)$  and  $\nabla \phi(x)$  are bounded. It is well known, see e.g. [19] and [8] for the infinite dimensional extension, that under hypothesis 3.1 the BSDE (4.4) admits a unique solution  $(Y_\tau^{t,x}, Z_\tau^{t,x}) \in \mathbb{K}_{cont}([0, T])$  and, if we further assume 4.1, setting  $v(t, x) := Y_t^{t,x}$ , it turns out that  $v$  is the unique mild solution of equation (4.1), and  $\nabla^B v(t, x) = Z_t^{t,x}$ . This leads immediately to the following lemma:

**Lemma 4.2** Assume that hypotheses 2.1 and 3.1 hold true. Let  $\psi_n$ ,  $\phi_n$  and  $l_n(\tau, \cdot)$  the inf-sup convolution of  $\psi$ ,  $\phi$  and  $l$  respectively. Then the Kolmogorov equation

$$\begin{cases} -\frac{\partial v}{\partial t}(t, x) = \mathcal{L}v(t, x) + \psi_n(\nabla^B v(t, x)) + l_n(t, x), & t \in [0, T], x \in H \\ v(T, x) = \phi_n(x), \end{cases} \quad (4.5)$$

admits a unique mild solution, with bounded Gâteaux derivative.

**Proof.** We recall, see e.g. [12] and [5], that the inf-sup convolution  $\phi_n$ ,  $\psi_n$  and  $l_n$  are defined respectively by

$$\phi_n(x) = \sup_{z \in H} \left\{ \inf_{y \in H} \left[ \phi(y) + n \frac{|z - y|_H^2}{2} \right] - n |x - z|_H^2 \right\}, \quad (4.6)$$

by

$$\psi_n(z) = \sup_{x \in \Xi} \left\{ \inf_{y \in \Xi} \left[ \psi(y) + n \frac{|x - y|_\Xi^2}{2} \right] - n |x - z|_\Xi^2 \right\}, \quad (4.7)$$

and by

$$l_n(t, x) = \sup_{z \in H} \left\{ \inf_{y \in H} \left[ l(t, y) + n \frac{|z - y|_H^2}{2} \right] - n |x - z|_H^2 \right\}, \quad (4.8)$$

for every  $t \in [0, T]$ . It turns out that  $\phi_n, l_n(t, \cdot) \in \mathcal{G}(H)$  and  $\psi_n \in \mathcal{G}(\Xi)$ , and so by [8], there exists a unique solution to the forward backward system

$$\begin{cases} dX_\tau = AX_\tau d\tau + BF(\tau, X_\tau) d\tau + BdW_\tau, & \tau \in [t, T] \subset [0, T], \\ X_t = x, \\ dY_\tau^n = -\psi_n(Z_\tau^n) d\tau - l_n(\tau, X_\tau) d\tau + Z_\tau^n dW_\tau, \\ Y_T^n = \phi_n(X_T), \end{cases} \quad (4.9)$$

that we denote by  $(X^{t,x}, Y^{n,t,x}, Z^{n,t,x})$  and by setting  $v_n(t, x) = Y_t^{n,t,x}$ , it turns out that  $v_n$  is the unique mild solution to the Kolmogorov equation (4.5).  $\square$

Next we prove that equation (4.1) admits a unique mild solution, in the case when the final conditions is lipschitz continuous,  $\psi$  is assumed further to be differentiable and  $l = 0$ , and that this solution is given in terms of the solution of the related FBSDE (4.3).

**Theorem 4.3** *Assume that hypotheses 2.1 and 3.1 hold true, assume that in equation (4.1)  $l = 0$  and  $\psi$  is Gâteaux differentiable. Then equation (4.1) admits a unique mild solution  $v(t, x)$  according to definition (4.1), and  $v(t, x)$  is given by*

$$v(t, x) := Y_t^{t,x}$$

with  $Y^{t,x}$  solution to the BSDE in (4.3).

**Proof.** We build the solution  $v$  by an approximation procedure. We let  $(\phi_n)_n$  be the inf-sup convolutions of  $\phi$ , see (4.6), and we consider the Kolmogorov equation

$$\begin{cases} -\frac{\partial v}{\partial t}(t, x) = \mathcal{L}v(t, x) + \psi(\nabla^B v(t, x)), & t \in [0, T], x \in H \\ v(T, x) = \phi_n(x), \end{cases} \quad (4.10)$$

whose mild solution  $v_n$  by lemma 4.2 exists and is unique, and it is given by

$$v_n(t, x) := Y_t^{n,t,x}$$

where  $(X^{t,x}, Y^{n,t,x}, Z^{n,t,x})$  solve the FBSDE

$$\begin{cases} dX_\tau = AX_\tau d\tau + BdW_\tau, & \tau \in [t, T] \subset [0, T], \\ X_t = x, \\ dY_\tau^{n,t,x} = -\psi(Z_\tau^{n,t,x}) d\tau + Z_\tau^{n,t,x} dW_\tau, \\ Y_T^{n,t,x} = \phi_n(X_T). \end{cases} \quad (4.11)$$



Let us set, for  $h \in \Xi$ ,  $F^{n,t,x,h} := \langle \nabla_x Y^{n,t,x}, Bh \rangle$  and  $V^{n,t,x,h} := \langle \nabla_x Z^{n,t,x}, Bh \rangle$ . The pair of processes  $(F^{n,t,x,h}, V^{n,t,x,h})$  solve the following BSDE

$$\begin{cases} dF_\tau^{n,t,x,h} = -\nabla\psi(Z_\tau^{n,t,x,h})V_\tau^{n,t,x,h} d\tau + V_\tau^{n,t,x,h} dW_\tau, \\ F_T^{n,t,x,h} = \nabla\phi_n(X_T^{t,x})e^{(T-t)A}Bh, \end{cases} \quad (4.12)$$

It is standard to see that

$$Y^{n,t,x} \rightarrow Y^{t,x} \text{ in } L^2(\Omega, C([0, T])), \quad Z^{n,t,x} \rightarrow Z^{t,x} \text{ in } L^2(\Omega \times [0, T]).$$

and so, by taking a subsequence  $n_k$ , also

$$Z^{n_k,t,x} \rightarrow Z^{t,x} dt \times d\mathbb{P} \text{ a.e. .}$$

For what concerns  $F^{n,t,x}$  and  $V^{n,t,x}$ , since by the lipschitzianity of  $\phi$  we have that  $\nabla\phi_n$  is bounded uniformly with respect to  $n$ , by assumptions 2.1 and 3.1 on  $\psi$ , by standard results on BSDEs we have that there exists a constant  $C$  independent on  $n$  such that

$$\|F^{n,t,x}\|_{L^2(\Omega, C([0, T]))} + \|V^{n,t,x}\|_{L^2(\Omega \times [0, T])} \leq C. \quad (4.13)$$

Next we investigate the convergence of  $F^{n,t,x,h}$ . By writing the BSDE satisfied by  $F^{n,t,x,h} - F^{j,t,x,h}$ ,  $n, j \geq 1$ , integrating between  $t$  and  $T$  and taking expectation, we get

$$\begin{aligned} F_t^{n,t,x,h} - F_t^{j,t,x,h} &= \mathbb{E} \left[ \nabla\phi_n(X_T^{t,x}) - \nabla\phi_j(X_T^{t,x}) \right] e^{(T-t)A} Bh \\ &\quad + \mathbb{E} \int_t^T \left( \nabla\psi(Z_\tau^{n,t,x})V_\tau^{n,t,x,h} - \nabla\psi(Z_\tau^{j,t,x})V_\tau^{j,t,x,h} \right) d\tau \end{aligned}$$

and consequently

$$\begin{aligned} |F_t^{n,t,x,h} - F_t^{j,t,x,h}| &\leq |\mathbb{E} \left[ \nabla\phi_n(X_T^{t,x}) - \nabla\phi_j(X_T^{t,x}) \right] e^{(T-t)A} Bh| \\ &\quad + |\mathbb{E} \int_t^T \left( \nabla\psi(Z_\tau^{n,t,x})V_\tau^{n,t,x,h} - \nabla\psi(Z_\tau^{j,t,x})V_\tau^{j,t,x,h} \right) d\tau| = I + II \end{aligned}$$

We start by estimating  $I$ :

$$\begin{aligned} I &= |\mathbb{E} \langle \nabla^B (\phi_n(X_T^{t,x}) - \phi_j(X_T^{t,x})), h \rangle| = |\nabla^B \langle \mathbb{E} (\phi_n - \phi_j)(X_T^{t,x}), h \rangle| \\ &= |\langle \nabla^B P_{t,T}[\phi_n - \phi_j](x), h \rangle| \leq c(T-t)\|\phi_n - \phi_j\|_\infty |h| \end{aligned}$$

In order to estimate  $II$ , we notice that since for every  $n \geq 1$  the final datum  $\phi$  and the hamiltonian function  $\psi$  are differentiable, then we have the following identification for  $F_t^{n,t,x}$ :

$$\langle \nabla^B Y_t^{n,t,x}, h \rangle := F_t^{n,t,x,h} = \langle \nabla^B v_n(t, x), h \rangle = \langle Z_t^{n,t,x}, h \rangle, \quad (4.14)$$

where in mild form  $v_n$  satisfies the integral equation

$$v_n(t, x) = P_{t,T}[\phi_n](x) + \int_t^T P_{t,s}[\psi(\nabla^B v_n(s, \cdot))](x) ds.$$

By taking the directional derivative  $\nabla^B$  we get

$$\nabla^B v_n(t, x) = \nabla^B P_{t,T}[\phi_n](x) + \int_t^T \nabla^B P_{t,s}[\psi(\nabla^B v_n(s, \cdot))](x) ds$$

So  $II$  can be rewritten in different ways as

$$\begin{aligned}
II &= |\mathbb{E} \int_t^T \left( \nabla \psi(Z_\tau^{n,t,x}) V_\tau^{n,t,x,h} - \nabla \psi(Z_\tau^{j,t,x}) V_\tau^{j,t,x,h} \right) d\tau| \\
&= |\mathbb{E} \int_t^T \langle \nabla^B (\psi(Z_\tau^{n,t,x}) - \psi(Z_\tau^{j,t,x})), h \rangle d\tau| \\
&= |\langle \nabla^B \int_t^T \mathbb{E} (\psi(Z_\tau^{n,t,x}) - \psi(Z_\tau^{j,t,x})) d\tau, h \rangle| \\
&= |\int_t^T \langle \nabla^B P_{t,\tau} [\psi(\nabla^B v_n(t, \cdot)) - \psi(\nabla^B v_j(t, \cdot))] , h \rangle d\tau|
\end{aligned}$$

We fix  $0 < \delta < T - t$  which will be chosen in the following and in order to estimate  $II$  we start by estimating

$$|\mathbb{E} \int_{t+\delta}^T \left( \nabla \psi(Z_\tau^{n,t,x}) V_\tau^{n,t,x,h} - \nabla \psi(Z_\tau^{j,t,x}) V_\tau^{j,t,x,h} \right) d\tau|$$

and we follow the calculations in [13], lemma 3.4. So

$$\begin{aligned}
&|\mathbb{E} \int_{t+\delta}^T \left( \nabla \psi(Z_\tau^{n,t,x}) V_\tau^{n,t,x,h} - \nabla \psi(Z_\tau^{j,t,x}) V_\tau^{j,t,x,h} \right) d\tau| \\
&= |\int_{t+\delta}^T \langle \nabla^B P_{t,\tau} [\psi(\nabla^B v_n(t, \cdot)) - \psi(\nabla^B v_j(t, \cdot))] , h \rangle d\tau| \\
&= |\int_{t+\delta}^T \int_H \left( \psi \left( \nabla^B v_n(s, y + e^{(s-t)A} x) \right) - \psi \left( \nabla^B v_j(s, y + e^{(s-t)A} x) \right) \right) \\
&\quad \left\langle Q_{s-t}^{-1/2} e^{(s-t)A} B h, Q_{s-t}^{-1/2} y \right\rangle \mathcal{N}(0, Q_{s-t})(dy) ds| \\
&\leq \int_{t+\delta}^T \left( \int_H |\psi \left( \nabla^B v_n(s, y + e^{(s-t)A} x) \right) - \psi \left( \nabla^B v_j(s, y + e^{(s-t)A} x) \right)|^2 \mathcal{N}(0, Q_{s-t}) \right)^{\frac{1}{2}} \\
&\quad \left( \int_H \left| \left\langle Q_{s-t}^{-1/2} e^{(s-t)A} B h, Q_{s-t}^{-1/2} y \right\rangle \right|^2 \mathcal{N}(0, Q_{s-t})(dy) \right)^{\frac{1}{2}} ds
\end{aligned}$$

where the last passage follows by the Cauchy-Schwartz inequality. So, by the Lipschitz property of  $\psi$ , and by the estimate (2.3), we get

$$\begin{aligned}
&|\mathbb{E} \int_{t+\delta}^T \left( \nabla \psi(Z_\tau^{n,t,x}) V_\tau^{n,t,x,h} - \nabla \psi(Z_\tau^{j,t,x}) V_\tau^{j,t,x,h} \right) d\tau| \\
&\leq C \int_{t+\delta}^T c(s-t) \left( \int_H |\nabla^B v_n(s, y + e^{(s-t)A} x) - \nabla^B v_j(s, y + e^{(s-t)A} x)|^2 \mathcal{N}(0, Q_{s-t}) \right)^{\frac{1}{2}} \\
&\leq C c(\delta) \mathbb{E} \int_{t+\delta}^T |Z_s^{n,t,x} - Z_s^{j,t,x}|^2 ds.
\end{aligned}$$

where in the last passage we have used the identification (4.14) of  $Z^{n,t,x}$  with  $\nabla^B v_n$ , and the fact that  $c(\cdot)$  is monotone not increasing. Now since  $Z^{n,t,x}$  is a Cauchy sequence in  $L^2(\Omega \times [0, T])$ , for every  $\varepsilon > 0$  it is possible to choose  $n, j \geq \bar{n}$  such that

$$\mathbb{E} \int_t^T |Z_s^{n,t,x} - Z_s^{j,t,x}|^2 ds \leq \frac{\varepsilon}{C c(\delta)}.$$

We can conclude that, choosen  $\varepsilon > 0$

$$|\mathbb{E} \int_{t+\delta}^T \left( \nabla \psi(Z_\tau^{n,t,x}) V_\tau^{n,t,x,h} - \nabla \psi(Z_\tau^{j,t,x}) V_\tau^{j,t,x,h} \right) d\tau| \leq \varepsilon, \quad (4.15)$$

independently on the choice of  $\delta$ . We now estimate

$$|\mathbb{E} \int_t^{t+\delta} \left( \nabla \psi(Z_\tau^{n,t,x}) V_\tau^{n,t,x,h} - \nabla \psi(Z_\tau^{j,t,x}) V_\tau^{j,t,x,h} \right) d\tau|.$$

We notice that by the Lipschitzianity of  $\psi$  the derivative  $\nabla \psi$  is bounded, and that  $V^{j,t,x,h}$  and  $V^{n,t,x,h}$  are uniformly bounded in  $L^2(\Omega \times [0, T])$ , so for any  $\varepsilon > 0$  we can choose  $\delta$  such that

$$|\mathbb{E} \int_t^{t+\delta} \left( \nabla \psi(Z_\tau^{n,t,x}) V_\tau^{n,t,x,h} - \nabla \psi(Z_\tau^{j,t,x}) V_\tau^{j,t,x,h} \right) d\tau| \leq \varepsilon.$$

and so we can conclude that for  $n, j \geq \bar{n}$

$$II \leq 2\varepsilon.$$

We have proved that

$$|F_t^{n,t,x,h} - F_t^{j,t,x,h}| \leq c(T-t) \|\phi_n - \phi_j\| |h| + \varepsilon$$

So by the identification in (4.14) we get that  $\nabla^B v_n$  is a Cauchy sequence in the space of the operators  $C_{c(\cdot)([0,T] \times H, \Xi^*)}^s$  and so it converges, to a limit that we denote by  $L(t, x)$ . We prove that  $L(t, x)$  coincides with  $\nabla^B v(t, x)$ . We have, for all  $t \in [0, T]$ ,  $x \in H$  and for all  $h \in \Xi$

$$\begin{aligned} \frac{v_n(t, x + sBh) - v_n(t, x)}{s} &= \int_0^1 \langle \nabla v_n(t, x + \alpha sBh), Bh \rangle d\alpha \\ &= \int_0^1 \langle \nabla^B v_n(t, x + \alpha sBh), h \rangle d\alpha. \end{aligned}$$

Passing to the limit as  $s \rightarrow 0$  on both sides we get

$$\langle \nabla^B v_n(t, x), h \rangle = L(t, x)h,$$

and the proof is concluded.  $\square$

**Remark 4.4** *It is immediate to see that  $\nabla^B v(t, x)$  is bounded. In fact we have seen in the proof of Theorem 4.3 that  $\nabla^B v(t, x)$  is the pointwise limit of  $\nabla^B v_n(t, x)$ , that  $\nabla^B v_n(t, x) = F_t^{n,t,x}$  and  $F_t^{n,t,x}$  is bounded, uniformly with respect to  $n$ . So we deduce that that  $\nabla^B v(t, x)$  is bounded. The same holds true for the solution whose existence and uniqueness is proved in Theorem 4.5 under more general assumptions than in Theorem 4.3.*

In Theorem 4.3 we have proved that the Kolmogorov equation admits a unique mild solution, assuming the running cost  $l = 0$  and  $\psi$  Gâteaux differentiable. The result is still true in the general context of the present paper, that is removing these additional assumptions.

**Theorem 4.5** *Assume that hypotheses 2.1 and 3.1 hold true. Then equation (4.1) admits a unique mild solution  $v(t, x)$  according to definition (4.1), and  $v(t, x)$  is given by*

$$v(t, x) := Y_t^{t,x}$$

*with  $Y^{t,x}$  solution to the BSDE in (4.3).*

**Proof.** The case of  $l \neq 0$  and lipschitz continuous with respect to  $x$ , uniformly with respect to  $t$ , can be handled as we have treated the final datum  $\phi$ . So we take  $l = 0$  and we look at removing the differentiability assumption on the Hamiltonian function  $\psi$ . We approximate  $\phi$  and  $\psi$  with their inf-sup convolutions, defined respectively in (4.6) and in (4.7). By Lemma 4.2, the approximating Kolmogorov equation

$$\begin{cases} -\frac{\partial v}{\partial t}(t, x) = \mathcal{L}v(t, x) + \psi_n(\nabla^B v(t, x)) + l(t, x), & t \in [0, T], x \in H \\ v(T, x) = \phi_n(x), \end{cases}$$

admits a unique mild solution  $v_n$  satisfying

$$v_n(t, x) = (P_{t,T}[\phi_n])(x) + \int_t^T P_{t,s}[\psi_n(\nabla^B v_n(s, \cdot))](x) ds.$$

Again by Lemma 4.2  $v_n$  is given by

$$v_n(t, x) := Y_t^{n,t,x}$$

where  $(X^{t,x}, Y^{n,t,x}, Z^{n,t,x})$  solve the FBSDE

$$\begin{cases} dX_\tau = AX_\tau d\tau + BdW_\tau, & \tau \in [t, T] \subset [0, T], \\ X_t = x, \\ dY_\tau^{n,t,x} = -\psi_n(Z_\tau^{n,t,x}) d\tau + Z_\tau^{n,t,x} dW_\tau, \\ Y_T^{n,t,x} = \phi_n(X_T). \end{cases} \quad (4.16)$$

Let us set, for  $h \in \Xi$ ,  $F^{n,t,x,h} := \langle \nabla_x Y^{n,t,x}, Bh \rangle$  and  $V^{n,t,x,h} := \langle \nabla_x Z^{n,t,x}, Bh \rangle$ . The pair of processes  $(F^{n,t,x,h}, V^{n,t,x,h})$  solve the following BSDE

$$\begin{cases} dF_\tau^{n,t,x,h} = -\nabla \psi_n(Z_\tau^{n,t,x,h}) V_\tau^{n,t,x,h} d\tau + V_\tau^{n,t,x,h} dW_\tau, \\ F_T^{n,t,x,h} = \nabla \phi_n(X_T^{t,x}) e^{(T-t)A} Bh, \end{cases} \quad (4.17)$$

The convergence of  $(Y^{n,t,x}, Z^{n,t,x})$  to  $(Y^{t,x}, Z^{t,x})$  stated in the proof of the previous Theorem 4.3 still holds true, as well as estimates (4.13). In order to investigate the convergence of  $F^{n,t,x,h}$ , we get

$$\begin{aligned} F_t^{n,t,x,h} - F_t^{j,t,x,h} &= \mathbb{E} \left[ \nabla \phi_n(X_T^{t,x}) - \nabla \phi_j(X_T^{t,x}) \right] e^{(T-t)A} Bh \\ &\quad + \mathbb{E} \int_t^T \left( \nabla \psi_n(Z_\tau^{n,t,x}) V_\tau^{n,t,x,h} - \nabla \psi_j(Z_\tau^{j,t,x}) V_\tau^{j,t,x,h} \right) d\tau. \end{aligned}$$

and consequently

$$\begin{aligned} |F_t^{n,t,x,h} - F_t^{j,t,x,h}| &\leq |\mathbb{E} [\nabla \phi_n(X_T^{t,x}) - \nabla \phi_j(X_T^{t,x})] e^{(T-t)A} Bh| \\ &\quad + |\mathbb{E} \int_t^T (\nabla \psi_n(Z_\tau^{n,t,x}) V_\tau^{n,t,x,h} - \nabla \psi_n(Z_\tau^{j,t,x}) V_\tau^{j,t,x,h}) d\tau| \\ &\quad + |\mathbb{E} \int_t^T (\nabla \psi_n(Z_\tau^{j,t,x}) V_\tau^{j,t,x,h} - \nabla \psi_j(Z_\tau^{j,t,x}) V_\tau^{j,t,x,h}) d\tau| = I + II + III \end{aligned}$$

The terms  $I$  and  $II$  can be treated as in the proof of Theorem 4.3, since, by the lipschitz character of  $\psi$  every  $\psi_n$  is Lipschitz continuous with the same Lipschitz constant as  $\psi$ , and the derivative  $\nabla \psi_n$  is bounded, uniformly with respect to  $n$ . It remains to estimate  $III$ : similarly

to what we have done in the proof of Theorem 4.3, we fix  $0 < \delta < T - t$  which will be chosen in the following and in order to estimate *III* we start by estimating

$$\begin{aligned} & |\mathbb{E} \int_{t+\delta}^T \left( \nabla \psi_n(Z_\tau^{j,t,x}) V_\tau^{j,t,x,h} - \nabla \psi_j(Z_\tau^{j,t,x}) V_\tau^{j,t,x,h} \right) d\tau| \\ &= |\mathbb{E} \int_{t+\delta}^T \int_H \left( \psi_n \left( v_j(s, y + e^{(s-t)A} x) \right) - \psi_j \left( v_j(s, y + e^{(s-t)A} x) \right) \right) \\ & \quad \left\langle Q_{s-t}^{-1/2} e^{(s-t)A} B h, Q_{s-t}^{-1/2} y \right\rangle \mathcal{N}(0, Q_{s-t})(dy) ds|. \end{aligned}$$

By the dominated convergence Theorem and by the uniform convergence of the sequence  $(\psi_n)_n$  this integral goes to 0 for  $n, j \rightarrow \infty$ . We can conclude that, choosen  $\varepsilon > 0$

$$|\mathbb{E} \int_{t+\delta}^T \left( \nabla \psi(Z_\tau^{n,t,x}) V_\tau^{n,t,x,h} - \nabla \psi(Z_\tau^{j,t,x}) V_\tau^{j,t,x,h} \right) d\tau| \leq \varepsilon, \quad (4.18)$$

independently on the choice of  $\delta$ . We now estimate

$$|\mathbb{E} \int_t^{t+\delta} \left( \nabla \psi_n(Z_\tau^{j,t,x}) V_\tau^{j,t,x,h} - \nabla \psi_j(Z_\tau^{j,t,x}) V_\tau^{j,t,x,h} \right) d\tau|.$$

Since  $\nabla \psi_n$  and  $\nabla \psi_j$  are uniformly bounded, and  $V_\tau^{j,t,x,h}$  and  $V_\tau^{n,t,x,h}$  are uniformly bounded in  $L^2(\Omega \times [0, T])$ , so for any  $\varepsilon > 0$  we can choose  $\delta$  such that

$$|\mathbb{E} \int_t^{t+\delta} \left( \nabla \psi(Z_\tau^{n,t,x}) V_\tau^{n,t,x,h} - \nabla \psi(Z_\tau^{j,t,x}) V_\tau^{j,t,x,h} \right) d\tau| \leq \varepsilon.$$

and so we can conclude that for  $n, j \geq \bar{n}$

$$III \leq 2\varepsilon.$$

We have proved that

$$|F_t^{n,t,x,h} - F_t^{j,t,x,h}| \leq c(T-t) \|\phi_n - \phi_j\| \|h\| + \varepsilon.$$

Now the proof goes on as the proof of Theorem 4.3.  $\square$

As a byproduct of the previous proof we have the following identification of  $Z_t^{t,x}$  with  $\nabla^B v(t, x)$ .

**Corollary 4.6** *Let  $(Y^{t,x}, Z^{t,x})$  be the solution of the BSDE in the FBSDE (4.3), assume that hypotheses 2.1 and 3.1 hold true and let  $v$  be mild solution of the semilinear Kolmogorov equation (4.1). Then*

$$\nabla^B v(\tau, X_\tau^{t,x}) = Z_\tau^{t,x}, \quad \mathbb{P} - a.s. \text{ for almost all } t \in [0, T]. \quad (4.19)$$

**Proof.** Let  $v_n$  be the solution of the approximating Kolmogorov equation (4.5) and let  $(Y^{n,t,x}, Z^{n,t,x})$  be the solution of the BSDE in the FBSDE (4.9). We know that for all  $t \in [0, T]$ ,  $x \in H$  the following identification holds true:

$$\nabla^B v_n(t, x) = Z_t^{n,t,x}, \text{ for almost all } t \in [0, T], x \in H.$$

Since  $\nabla^B v_n(t, x) \rightarrow \nabla^B v(t, x)$  for all  $t \in [0, T]$  and  $x \in H$ , and  $\nabla^B v_n(t, x)$  is bounded uniformly with respect to  $n$ , we get that  $\nabla^B v_n(\tau, X_\tau^{t,x}) \rightarrow \nabla^B v(\tau, X_\tau^{t,x})$  in  $L^2(\Omega \times [0, T])$ , and so we get the identification (4.19).  $\square$

**Remark 4.7** We show how to handle the case of a semilinear Kolmogorov equation (4.1) where  $\mathcal{L}$  is the generator of a perturbed Ornstein-Uhlenbeck process as (2.5), that is

$$(\mathcal{L}_t f)(x) = \frac{1}{2}(\text{Tr} B B^* \nabla^2 f)(x) + \langle Ax + BF(t, x), \nabla f(x) \rangle.$$

We have already noticed in remark 2.4 that if  $A$  and  $B$  satisfy hypothesis 2.1 and if  $F$  is sufficiently regular, then the perturbed Ornstein-Uhlenbeck transition semigroup satisfies the regularizing property stated in Lemma 2.3.

We do not use this property for the solution of the semilinear Kolmogorov equation, but an equivalent representation of the mild solution in terms of an Ornstein-Uhlenbeck transition semigroup based on the Girsanov transform. To this aim, notice that, at least in the case of  $\phi$  and  $\psi$  differentiable, we can apply the Girsanov theorem in the forward-backward system

$$\begin{cases} dX_\tau = AX_\tau d\tau + BF(\tau, X_\tau) d\tau + BdW_\tau, & \tau \in [t, T], \\ X_\tau = x, & \tau \in [0, t], \\ dY_\tau^{t,x} = -\psi(Z_\tau^{t,x}) - l(\tau, X_\tau^{t,x}) d\tau + Z_\tau^{t,x} dW_\tau, & \tau \in [0, T], \\ Y_T^{t,x} = \phi(X_T^{t,x}), \end{cases}$$

or we can follow [10]. We get that the mild solution of equation (4.1) can be represented, for all  $t \in [0, T]$ ,  $x \in H$ , as

$$\begin{aligned} v(t, x) &= R_{t,T}[\phi](x) + \int_t^T R_{t,s}[l(s, \cdot)](x) ds \\ &\quad + \int_t^T R_{t,s}[\psi(\nabla^B v(s, \cdot))](x) ds + \int_t^T R_{t,s}[\nabla^B v(s, \cdot) F(s, \cdot)](x) ds. \end{aligned}$$

Here  $(R_{t,T})_{t \in [0, T]}$  is the transition semigroup of the corresponding Ornstein-Uhlenbeck process

$$\begin{cases} dX_\tau = AX_\tau d\tau + BdW_\tau, & \tau \in [t, T], \\ X_t = x, & \tau \in [0, t]. \end{cases}$$

The new Hamiltonian function is given by

$$\tilde{\psi}(t, x, z) := \psi(z) + zF(t, x) \quad (4.20)$$

and by a straightforward generalization of Theorem 4.5 the case of a perturbed Ornstein-Uhlenbeck process is covered.  $\square$

## 5 The semilinear Kolmogorov equation in the Banach space framework

In this Section we revisit the results contained in the previous sections in the case when the problem is considered in a Banach space  $E$  instead of the Hilbert space  $H$ . For the Banach space framework, we mainly refer to the papers [14], [15] and [16]. From now we assume that  $E$  admits a countable Schauder basis.

From now on, we consider a Banach space  $E$  continuously and densely embedded in a real and separable Hilbert space  $H$ . We have to adequate the Hilbert space case to this new context. In general, we can not guarantee that  $B(\Xi) \subset E$ . We make the following assumption which is verified in most of the applications.

**Hypothesis 5.1** *There exists a subspace  $\Xi_0$  dense in  $\Xi$  such that  $B(\Xi_0) \subset E$*

In the following we extend the definition of  $B$  differentiability to continuous functions  $f : E \rightarrow \mathbb{R}$ , as in [14], definition 2.7.

**Definition 5.1** *For a map  $f : E \rightarrow \mathbb{R}$  the  $B$ -directional derivative  $\nabla^B f$  at a point  $x \in E$  in direction  $\xi \in \Xi_0$  is defined as usual. We say that a continuous function  $f$  is  $B$ -Gateaux differentiable at a point  $x \in E$  if  $f$  admits the  $B$ -directional derivative  $\nabla^B f(x; \xi)$  in every directions  $\xi \in \Xi_0$  and there exists a linear operator  $\nabla^B f(x)$  from  $\Xi_0$  with values in  $\mathbb{R}$ , such that  $\nabla^B f(x; \xi) = \nabla^B f(x) \xi$  and  $|\nabla^B f(x) \xi| \leq C_x \|\xi\|_{\Xi}$ , where  $C_x$  does not depend on  $\xi$ . So the operator  $\nabla^B f(x)$  can be extended to the whole  $\Xi$ , and we denote this extension again by  $\nabla^B f(x)$ , the  $B$ -gradient of  $f$  at  $x$ . The definition of  $f$   $B$ -Gâteaux differential and of the class  $\mathcal{G}^B(E)$  of functions  $f \in C_b(E)$  that are  $B$ -Gâteaux differentiable follows in the usual natural way, see [14].*

We consider an Ornstein-Uhlenbeck process in  $E$ , that is a Markov process  $X$  solution to equation

$$\begin{cases} dX_\tau = AX_\tau d\tau + BdW_\tau, & \tau \in [t, T] \\ X_t = x, \end{cases} \quad (5.1)$$

On the operator  $A$  and  $B$  we make the following assumptions:

**Hypothesis 5.2** *We assume that  $A$  generates a  $C_0$  semigroup in  $E$ ; and we suppose that there exists  $\omega \in \mathbb{R}$  such that  $\|e^{tA}\|_{L(E,E)} \leq e^{\omega t}$ , for all  $0 \leq t \leq T$ . We assume that  $e^{tA}$ ,  $t \geq 0$ , admits an extension to a  $C_0$  semigroup of bounded linear operators in  $H$ , whose generator is denoted by  $A_0$  or by  $A$  if no confusion is possible. The operator  $B \in L(\Xi, H)$  is such that the operators*

$$Q_\tau = \int_0^\tau e^{sA_0} B B^* e^{sA_0^*} ds, \quad \tau \geq 0,$$

*are of trace class for every  $\tau \in [0, T]$ , so that the stochastic convolution  $W_A(\tau)$  is well defined in  $H$ . We assume further that  $W_A(\tau)$  admits an  $E$ -continuous version.*

We make the following assumptions on  $A$  and  $B$ , which imply  $B$ -regularizing properties on the Ornstein-Uhlenbeck transition semigroup  $P_t$ .

**Hypothesis 5.3** *The operators  $A$  and  $B$  are such that*

$$e^{tA_0} B(\Xi) \subset Q_t^{1/2}(H). \quad (5.2)$$

*and*

$$\left\| Q_t^{-1/2} e^{tA_0} B \right\| \leq c(t), \text{ for } 0 < t \leq T. \quad (5.3)$$

*with  $c(t)$  as in Hypothesis 2.2.*

Under hypothesis 5.3 it is possible to prove that the transition semigroup  $P_t$  enjoys the  $B$ -regularizing property, with an analogous of Lemma 2.3, following also lemma 2.8 in [14]. Let us consider also a perturbed Ornstein-Uhlenbeck process in  $E$ :

$$\begin{cases} dX_\tau = AX_\tau d\tau + BF(X_\tau) d\tau + BdW_\tau, & \tau \in [t, T] \\ X_t = x. \end{cases} \quad (5.4)$$

In order to get the  $B$ -regularizing property for the transition semigroup of the perturbed Ornstein-Uhlenbeck process in  $E$ , we have further to assume:

**Hypothesis 5.4** For every  $\xi \in \Xi$  and for every  $t > 0$   $\|e^{tA_0} B\xi\|_E \leq ct^{-\alpha} \|\xi\|_\Xi$ , for some constant  $c > 0$  and  $0 < \alpha < \frac{1}{2}$ .

Then, with the nonlinear term  $F$  in (5.4) lipschitz continuous and differentiable, following [14], Section 4, the  $B$  regularizing property for the transition semigroup of the perturbed Ornstein-Uhlenbeck process can be proved, with blow up given by the function  $c(t)$  in hypothesis 5.3.

We now solve semilinear Kolmogorov equations in Banach spaces. Let  $\mathcal{L}_t$  be the generator of the transition Ornstein-Uhlenbeck transition semigroup  $P_t$  related to the  $E$ -valued process defined by (5.1), that is, at least formally,

$$(\mathcal{L}f)(x) = \frac{1}{2}(TrBB^*\nabla^2 f)(x) + \langle Ax, \nabla f(x) \rangle.$$

Let us consider the following equation

$$\begin{cases} -\frac{\partial v}{\partial t}(t, x) = \mathcal{L}v(t, x) + \psi(\nabla^B v(t, x)) + l(t, x), & t \in [0, T], x \in E \\ v(T, x) = \phi(x), \end{cases} \quad (5.5)$$

We can adequate immediately definition 4.2 in order to get the notion of mild solution of a semilinear Kolmogorov equation in  $E$ . Also hypothesis 3.1 can be immediately adequated to the Banach space framework by substituting the real and separable Hilbert space  $H$  with the Banach space  $E$ .

Moreover, the counterpart of Theorems 4.3 and 4.5 holds true, and it is summarized in the following Theorem.

**Theorem 5.5** Assume that hypotheses 5.1, 2.1 and 3.1, with the Banach space  $E$  in the place of the Hilbert space  $H$ , hold true. Then equation (4.1) admits a unique mild solution  $v(t, x)$  according to definition (4.1), and  $v(t, x)$  is given by

$$v(t, x) := Y_t^{t, x}$$

with  $Y^{t, x}$  solution to the BSDE in (4.3).

**Proof.** We give the sketch of the proof, in the case  $l = 0$  for the sake of simplicity. We decide not to smooth the final datum  $\phi$  by means of the analogous of inf-sup convolutions in Banach spaces, see [21], but to use the approximation procedure introduced in [20] for Hilbert spaces and that can be generalized to Banach spaces with a countable Schauder basis. Namely, we can set

$$\phi_n(x) = \int_{\mathbb{R}^n} \rho_n(y - Q_n x) \phi\left(\sum_{i=1}^n y_i e_i\right) dy, \quad (5.6)$$

where  $(e_i)_{i \geq 1}$  are the elements of the Schauder basis. We have that for all  $n \geq 1$ ,  $\phi_n$  is lipschitz continuous, with the same lipschitz constant as  $\phi$ , and it is Gâteaux differentiable, and

$$|\phi_n(x)| \leq \|\phi\|_\infty.$$

We notice that the approximation of  $\phi$  by means of  $\phi_n$  as defined in (5.6) is only pointwise:

$$\lim_{n \rightarrow \infty} \phi_n(x) = \phi(x), \quad \forall x \in E.$$

Moreover we smooth the Hamiltonian function  $\psi$ , which is still defined on the Hilbert space  $\Xi$  with its inf-sup convolution, see (4.7). We get an approximating Kolmogorov equation in the Banach space  $E$  given by

$$\begin{cases} -\frac{\partial v}{\partial t}(t, x) = \mathcal{L}v(t, x) + \psi_n(\nabla^B v(t, x)), & t \in [0, T], x \in E \\ v(T, x) = \phi_n(x). \end{cases} \quad (5.7)$$



For the approximating Kolmogorov equation (5.7) we can apply [15], Theorem 6.2, to get that by setting  $(Y^{n,t,x}, Z^{n,t,x})$  solution of the FBSDE (4.9) with the process  $X$  with values in  $E$ , the function  $v_n(t, x) := Y_t^{n,t,x}$  turns out to be a mild solution of the approximating Kolmogorov equation (5.7), for every  $n \in \mathbb{N}$ . We have to show that  $v_n(t, x), \nabla^B v_n(t, x)$  converge and that  $v(t, x) = \lim_{n \rightarrow \infty} v_n(t, x)$  is the mild solution of the Kolmogorov equation (4.1) in  $E$ . We let  $(Y^n, Z^n)$  be the solution of the approximating BSDE in the FBSDE in the Banach space framework

$$\begin{cases} dX_\tau = AX_\tau d\tau + BdW_\tau, & \tau \in [t, T] \subset [0, T], \\ X_t = x, \\ dY_\tau^n = -\psi_n(Z_\tau^n) d\tau + Z_\tau^n dW_\tau, \\ Y_T^n = \phi_n(X_T), \end{cases} \quad (5.8)$$

and  $(Y, Z)$  be the solution of the BSDE in the FBSDE in the Banach space framework

$$\begin{cases} dX_\tau = AX_\tau d\tau + BdW_\tau, & \tau \in [t, T] \subset [0, T], \\ X_t = x, \\ dY_\tau = -\psi(Z_\tau) d\tau + Z_\tau dW_\tau, \\ Y_T = \phi(X_T). \end{cases} \quad (5.9)$$

In order to show that the pair of processes  $(Y^n, Z^n)$  converge to the pair  $(Y, Z)$ , with  $\phi_n \rightarrow \phi$  only pointwise, we apply proposition 5.5 in [17]. It is fundamental that  $(\phi_n)_n$  are uniformly bounded. Next, following theorems 4.3 and 4.5, it is possible to show that  $v_n(t, x), \nabla^B v_n(t, x)$  converge and that  $v(t, x) = \lim_{n \rightarrow \infty} v_n(t, x)$  is the mild solution of the Kolmogorov equation (4.1) in  $E$ .  $\square$

As in Section 4 we can notice that  $\nabla^B v(t, x)$  is bounded, see Remark 4.4, and moreover as in Corollary 4.6, it is possible to show that

$$\nabla^B v(\tau, X_\tau^{t,x}) = Z_\tau^{t,x}, \mathbb{P} - \text{a.s.}, \text{ for a.a. } t \in [0, T].$$

Finally, following the outlines given in remark 4.7, and assuming also that hypothesis 5.4 holds true, it is possible to handle the case of the second order differential operator  $\mathcal{L}_t$  generator of a perturbed Ornstein-Uhlenbeck process in  $E$ , at least in the case of  $F$  lipschitz continuous with respect to  $x$ .

## 6 Solution of the optimal control problem

In this Section we consider the stochastic optimal control problem introduced in Section 3. This problem can be solved in the Hilbert space framework presented in Section 3, but we skip the solution of the optimal control problem in the Hilbert space framework and we consider it in the Banach space framework, that allows more generality on the choice of the costs, see in particular Section 6.1.

The controlled state equation takes its values in  $E$  and it is given by

$$\begin{cases} dX_\tau^u = [AX_\tau^u + Bu_\tau] d\tau + BdW_\tau, & \tau \in [t, T] \\ X_t^u = x. \end{cases} \quad (6.1)$$

The control  $u$  is an  $(\mathcal{F}_\tau)_\tau$ -predictable process with values in a closed and bounded set  $U$  of the Hilbert space  $\Xi$ . Thanks to Hypothesis 5.2 on the operators  $A$  and  $B$ , and to Hypotheses 5.1 and 5.4, the mild solution of the controlled state equation (6.1) is a well defined  $E$ -valued process.

Beside equation (6.1), define the cost

$$J(t, x, u) = \mathbb{E} \int_t^T [l(s, X_s^u) + g(u_s)] ds + \mathbb{E} \phi(X_T^u). \quad (6.2)$$

for real functions  $l$  on  $[0, T] \times E$ ,  $\phi$  on  $E$  and  $g$  on  $U$ , which satisfy Hypothesis 3.1 with  $E$  in the place of  $H$ .

The control problem will be solved by means of the dynamic programming principle, the value function of the optimal control problem introduced in Section 3 is identified with the solution of the HJB equation related. The HJB equation has the same structure of the semilinear Kolmogorov equation we have studied in Section 5, equation (5.5), with final datum  $\phi$  equal to the final cost, and with semilinear term  $\psi$  equal to the Hamiltonian function defined in (3.3).

**Theorem 6.1** *Assume Hypotheses 5.1, 5.2, 5.3, 5.4 and 3.1, adequated to the Banach space framework, hold true. Let  $v$  be the unique mild solution to equation (5.5). For every  $t \in [0, T]$ ,  $x \in H$  and for all admissible control  $u$  we have  $J(t, x, u) \geq v(t, x)$ , and the equality holds if and only if, for a.a.  $s \in [0, T]$ ,  $\mathbb{P}$ -a.s.*

$$u_s \in \Gamma(\nabla^B v(s, X_s^{u, t, x})).$$

**Proof.** At first we assume that the Hamiltonian function  $\psi$  is Lipschitz continuous and Gâteaux differentiable and we approximate the current cost  $l$ , the final cost  $\phi$  by means of the pointwise approximation introduced in [20] and that we have already recalled in the proof of Theorem 5.5, formula (5.6). With all the data smooth, we can apply Proposition 5.5, Corollary 5.6 and Theorem 5.7 in [15], in order to get

$$J_n(t, x, u) := \mathbb{E} \int_t^T [l_n(s, X_s^u) + g(u_s)] ds + \mathbb{E} \phi_n(X_T^u) \quad (6.3)$$

$$= v_n(t, x) - \int_t^T [\psi(\nabla^B v_n(s, X_s)) - g(u_s) - \nabla^B v_n(s, X_s) u_s] ds, \quad (6.4)$$

where  $v_n(t, x)$  is the mild solution of the approximating Kolmogorov equation

$$\begin{cases} -\frac{\partial v_n}{\partial t}(t, x) = \mathcal{L}v_n(t, x) + \psi(\nabla^B v_n(t, x)) + l_n(t, x), & t \in [0, T], x \in H \\ v_n(T, x) = \phi_n(x), \end{cases}$$

Letting  $n \rightarrow \infty$  in (6.3) we get

$$J(t, x, u) = v(t, x) - \int_t^T [\psi(\nabla^B v(s, X_s)) - g(u_s) - \nabla^B v(s, X_s) u_s] ds,$$

and by the definition of  $\psi$  the result follows.

If  $\psi$  is not Gâteaux differentiable, we approximate it by means of its inf-sup convolutions  $\psi_k$

defined in 4.7. We get, by Proposition 5.5 in [15] and by Corollary 4.6

$$J(t, x, u) = \mathbb{E} \int_t^T [l(s, X_s^u) + g(u_s)] ds + \mathbb{E} \phi(X_T^u) \quad (6.5)$$

$$= \mathbb{E} \int_t^T [l(s, X_s) + \psi_k(\nabla^B v_k(s, X_s))] ds + \mathbb{E} \phi(X_T) \quad (6.6)$$

$$+ \mathbb{E} \int_t^T [g(u_s) - \psi(\nabla^B v_k(s, X_s)) + \nabla^B v_k(s, X_s) u_s] ds \quad (6.7)$$

$$+ \mathbb{E} \int_t^T [\psi(\nabla^B v_k(s, X_s)) - \psi_k(\nabla^B v_k(s, X_s))] ds \quad (6.8)$$

$$= v_k(t, x) - \int_t^T [\psi(\nabla^B v(s, X_s)) - g(u_s) - \nabla^B v(s, X_s) u_s] ds \quad (6.9)$$

$$- \int_t^T [\psi_k(\nabla^B v_k(s, X_s)) - \psi(\nabla^B v_k(s, X_s))] ds \quad (6.10)$$

where  $v_k(t, x)$  is the mild solution of the approximating Kolmogorov equation

$$\begin{cases} -\frac{\partial v}{\partial t}(t, x) = \mathcal{L}v(t, x) + \psi_k(\nabla^B v(t, x)) + l(t, x), & t \in [0, T], x \in H \\ v(T, x) = \phi(x), \end{cases}$$

Letting  $k \rightarrow \infty$  in (6.5) we get

$$J(t, x, u) = v(t, x) - \int_t^T [\psi(\nabla^B v(s, X_s^u)) - g(u_s) - \nabla^B v(s, X_s^u) u_s] ds,$$

and the Theorem is proved.  $\square$

Moreover we can perform the synthesis of the optimal control. Let us define now the *optimal feedback law*:

$$u(\tau, x) = \gamma(\nabla^B v(\tau, X_\tau^{u, \tau, x})), \quad \tau \in [t, T], x \in H.$$

We consider the *closed loop equation*

$$\begin{cases} dX_\tau^u = [AX_\tau^u + B\gamma(\nabla^B v(\tau, \overline{X}_\tau))] d\tau + BdW_\tau, & \tau \in [t, T] \\ X_t^u = x. \end{cases} \quad (6.11)$$

and we assume that it admits a solution

$$\overline{X}_s = e^{(s-t)A} x_0 + \int_t^s e^{(r-t)A} B(\gamma(\nabla^B v(r, \overline{X}_r))) dr + \int_t^s e^{(r-t)A} B dW_r. \quad (6.12)$$

Then the pair  $(\overline{u} = u(s, \overline{X}_s), \overline{X}_s)_{s \in [t, T]}$  is optimal for the control problem. We recall that existence of a solution of the closed loop equation is not obvious, and that this problem can be avoided by formulating the optimal control problem in the weak sense, following [6]. The advantage of the weak formulation is that the closed loop equation is solvable in the weak sense by a Girsanov change of measure.

## 6.1 Optimal control problem for the heat equation

In this Section we treat stochastic optimal control problems related to stochastic heat equations with control and noise on a subdomain. Stochastic optimal control problems for reaction diffusion equations have been extensively studied in the literature. We cite in particular the papers

[2] and [3], where equations with a more general structure than equation (2.6) below are treated, but some more smoothing properties on the transition semigroup are required, and the paper [16] where the case of a non linear term in the heat equation is treated and the problem is solved in the Banach space of continuous functions, on the contrary all the coefficients are asked to be Gâteaux differentiable. In the present paper we are able to remove differentiability assumptions on the cost. Moreover we are able to consider only Lipschitz continuous current and final costs in the Banach space of continuous functions, so we are able to treat costs like the supremum of the state.

Applying the results in section 4, we can also solve the associated HJB equation. We are not able to solve the related HJB equation as in [13] by a suitable fixed point argument. Indeed the transition semigroup of the state equation is Strong Feller. This regularizing property is related to the fact that deterministic linear heat equations with control on a subinterval are null controllable, see e.g. [23], but the minimal energy blows up too fast, see e.g. [7] and [22]. So since the minimal energy blows up too fast at 0, namely it is not integrable near 0, fixed point arguments as the ones used in [10], [13] and [14] cannot be applied to this problem.

In the following we present the controlled heat equation we are able to treat, which is the controlled version of (2.6).

We denote by  $H$ , which in this case coincides with  $\Xi$ , the Hilbert space  $L^2([0, 1])$  and by  $E$  the space of continuous functions on  $[0, 1]$ . We consider the controlled heat equation

$$\begin{cases} \frac{\partial y}{\partial s}(s, \xi) = \Delta y(s, \xi) + 1_{\mathcal{O}}(\xi)u(s, \xi) + 1_{\mathcal{O}}(\xi)\frac{\partial W}{\partial s}(s, \xi), & s \in [t, T], \xi \in [0, 1], \\ y(t, \xi) = x(\xi), \\ \frac{\partial}{\partial \xi}y(s, \xi) = 0, & \xi = 0, 1. \end{cases} \quad (6.13)$$

Here  $\frac{\partial W}{\partial s}(s, \xi)$  is a space time white noise and  $u : [0, T] \times [0, 1] \rightarrow \mathbb{R}$  is the control process such that  $u \in L^2([0, T], L^2([0, 1]))$ .

We are able to treat cost functionals

$$J(t, x, u) = \mathbb{E} \int_t^T [l(s, X^u(\cdot)) + g(u_s)] ds + \mathbb{E} \phi(X^u(\cdot)), \quad (6.14)$$

where

$$l : [0, T] \times C([0, 1]) \rightarrow \mathbb{R}, \quad \phi : C([0, 1]) \rightarrow \mathbb{R}.$$

For example, we are able to handle the case of final and current cost given by the supremum of the state over the interval  $[0, 1]$ :

$$l(s, x(\cdot)) = \sup_{\xi \in [0, 1]} x(\xi), \quad \phi(x(\cdot)) = \sup_{\xi \in [0, 1]} x(\xi)$$

To fit our hypotheses 3.1 we have to suppose only Lipschitz continuity of  $\phi$  and of  $l$  with respect to  $x \in C([0, 1])$ .

**Hypothesis 6.2** *The costs  $l$  and  $\phi$  are all Borel measurable and real valued. Moreover*

1.  $l : [0, T] \times C([0, 1]) \rightarrow \mathbb{R}$  is continuous and bounded, and for all  $t \in [0, T]$ ,  $l(t, \cdot) : C([0, 1]) \rightarrow \mathbb{R}$  is Lipschitz continuous.
2.  $\phi : C([0, 1]) \rightarrow \mathbb{R}$  is continuous, bounded, and Lipschitz continuous.

The heat equation (2.6) can be written in abstract way in the Banach space  $E$  as

$$\begin{cases} dX_\tau^u = AX_\tau^u + Bu_\tau d\tau + BdW_\tau & \tau \in [t, T] \\ X_t^u = x_0, \end{cases} \quad (6.15)$$

where  $A$  is the Laplace operator in  $E = C([0, 1])$  with Neumann boundary conditions, and  $B$  is the multiplication by the indicator function  $1_{\mathcal{O}_0}$ . The operator  $A$  turns out to be the generator of a strongly continuous semigroup in  $E$ , with an extension to  $H$ . Also hypothesis 5.1 is satisfied by taking  $\Xi_0 = \{f \in C([0, 1]) : f(a) = f(b) = 0\}$ , where we recall that  $\mathcal{O} = [a, b]$ . With this choice,  $\Xi_0$  is dense in  $\Xi = L^2([0, 1])$ , and  $B(\Xi_0) \subset E = C([0, 1])$ . We refer also to [16] for more details in the reformulation.

We consider the Hamilton Jacobi Bellman equation relative to (6.15)

$$\begin{cases} -\frac{\partial v}{\partial t}(t, x) = \mathcal{L}v(t, x) + \psi(\nabla^B v(t, x)) + l(t, x), & t \in [0, T], x \in H, \\ u(T, x) = \phi(x), \end{cases} \quad (6.16)$$

where  $\psi$  is the Hamiltonian function defined in (3.3). In order to find mild solutions of the HJB equation and to solve the optimal control problem, we see that  $A$  and  $B$  are such that hypotheses 5.2, 5.1, 5.4 and 5.3 are satisfied. By hypotheses 6.2 on  $l$  and  $\phi$  in the definition, hypothesis 3.1 is satisfied.

**Theorem 6.3** *Assume that hypothesis 6.2 holds true. Then equation (6.16) has a unique mild solution  $v$  and for all the admissible controls  $J(t, x, u) \geq v(t, x)$ . Moreover  $J(t, x, u) = v(t, x)$  if and only if for a.a.  $s \in [0, T]$ ,  $\mathbb{P}$ -a.s.*

$$u_s \in \Gamma(\nabla^B v(s, X_s^{u, t, x})).$$

## References

- [1] J.P. Aubin, H. Frankowska, Set valued analysis, Birkhäuser, Boston, 1990.
- [2] Cerrai, S., Differentiability of Markov semigroups for stochastic reaction-diffusion equations and applications to control. Stochastic Proc. Appl. 1999, 83 (1), 15-37
- [3] Cerrai, S., Optimal control problems for stochastic, reaction-diffusion equations and applications to control. SIAM J. Contr. Optim. 2001, 39 (6), 1779-1816.
- [4] G. Da Prato and J. Zabczyk, Stochastic equations in infinite dimensions, Encyclopedia of Mathematics and its Applications 44, Cambridge University Press, 1992.
- [5] G. Da Prato and J. Zabczyk, Second order partial differential equations in Hilbert spaces. London Mathematical Society Note Series, 293, Cambridge University Press, Cambridge, 2002.
- [6] W. H. Fleming, H. M. Soner, Controlled Markov processes and viscosity solutions. Applications of Mathematics 25. Springer-Verlag, 1993.
- [7] Fernández-Cara, E., and Zuazua, E., The cost of approximate controllability for heat equations: the linear case. Adv. Differential Equations 2000, 5 (4-6), 465-514.
- [8] M. Fuhrman, G. Tessitore, Nonlinear Kolmogorov equations in infinite dimensional spaces: the backward stochastic differential equations approach and applications to optimal control. Ann. Probab. 30 (2002), no. 3, 1397–1465.

- [9] M. Fuhrman, G. Tessitore, Generalized directional gradients, backward stochastic differential equations and mild solutions of semilinear parabolic equations. *Appl. Math. Optim.* 51 (2005), no. 3, 279–332.
- [10] F. Gozzi, Regularity of solutions of second order Hamilton-Jacobi equations in Hilbert spaces and applications to a control problem, (1995) *Comm. Partial Differential Equations* 20, pp. 775–826.
- [11] F. Gozzi, Global regular solutions of second order Hamilton-Jacobi equations in Hilbert spaces with locally Lipschitz nonlinearities, (1996) *J. Math. Anal. Appl.* 198, pp. 399–443.
- [12] J. M. Lasry and P. L. Lions, A remark on regularization in Hilbert spaces, (1986) *Israel. J. Math.* 55, pp. 257–266.
- [13] F. Masiero, Semilinear Kolmogorov equations and applications to stochastic optimal control, *Appl. Math. Optim.*, 51 (2005), pp. 201–250.
- [14] F. Masiero, Regularizing properties for transition semigroups and semilinear parabolic equations in Banach spaces. *Electron. J. Probab.* 12 (2007), no. 13, 387–419
- [15] F. Masiero, Stochastic optimal control problems and parabolic equations in Banach spaces. *SIAM J. Control Optim.* 47 (2008), no. 1, 25–300.
- [16] F. Masiero, Stochastic optimal control for the stochastic heat equation with exponentially growing coefficients and with control and noise on a subdomain, *Stoch. Anal. Appl.* 26 (2008), no. 4, 877–902.
- [17] F. Masiero, A. Richou, HJB equations in infinite dimensions with locally Lipschitz Hamiltonian and unbounded terminal condition, *J. Differential Equations* 257 (2014), no. 6, 1989–2034.
- [18] E. Pardoux, S. Peng, Adapted solution of a backward stochastic differential equation, *Systems and Control Lett.* 14, 1990, 55–61.
- [19] E. Pardoux, S. Peng, Backward stochastic differential equations and quasilinear parabolic partial differential equations, in: *Stochastic partial differential equations and their applications*, eds. B.L. Rozowskii, R.B. Sowers, 200–217, *Lecture Notes in Control Inf. Sci.* 176, Springer, 1992.
- [20] S. Peszat and J. Zabczyk, *Strong Feller property and irreducibility for diffusions on Hilbert spaces*, *Ann. Probab.* 23 (1995), no. 1, 157–172.
- [21] T. Strömberg, On regularization in Banach spaces. *Ark. Mat.* 34 (1996), no. 2, 383–406.
- [22] Zuazua, E., *Some problems and results on the controllability of partial differential equations*. European Congress of Mathematics, Vol. II (Budapest, 1996), 276311, *Progr. Math.*, 169, Birkhäuser, Basel, 1998.
- [23] Zuazua, E., *Some results and open problems on the controllability of linear and semilinear heat equations*. Carleman estimates and applications to uniqueness and control theory (Cortona, 1999), 191–211, *Progr. Nonlinear Differential Equations Appl.*, 46, Birkhäuser Boston, Boston, MA, 2001.